# INTEGER TRIANGLES WITH FIXED PERIMETER

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**Theorem.** *The number of solutions for* $a+b+c=n$*, where* $a$*,* $b$*,* $c$ *and* $n$ *are integers with* $n$ *already given,* $a+b>c$ *and* $0<a\leqslant b\leqslant c$*, is:*

$$f(c)=\sum_{c=\lceil\frac{n}{3}\rceil}^{\lceil\frac{n}{2}\rceil-1}\left(1+c-\left\lceil\frac{n-c}{2}\right\rceil\right)$$

*Proof.* Since, $c\geqslant a$ and $c\geqslant b$,

$$\begin{aligned} c+c+c &\geqslant a+b+c \\ 3c &\geqslant n \\ c &\geqslant \frac{n}{3} \end{aligned}$$

Since, $c$ and $n$ are integers, the above inequality can be written as:

$$c\geqslant\left\lceil\frac{n}{3}\right\rceil \tag{1}$$

Since, $a+b>c$,

$$\begin{aligned} a+b+c &> c+c \\ n &> 2c \\ \frac{n}{2} &> c \end{aligned}$$

Since, $c$ and $n$ are integers, the above inequality can be written as:

$$c\leqslant\left\lceil\frac{n}{2}\right\rceil-1 \tag{2}$$

From (1) and (2), we get:

$$\left\lceil\frac{n}{3}\right\rceil\leqslant c\leqslant\left\lceil\frac{n}{2}\right\rceil-1 \tag{3}$$

Therefore, there are $\lceil \frac{n}{2} \rceil - \lceil \frac{n}{3} \rceil$ possible values for $c$. For each value of $c$, $b \leqslant a$. Therefore,

$$\begin{aligned} b + b + c &\geqslant a + b + c \\ 2b + c &\geqslant n \\ b &\geqslant \frac{n - c}{2} \end{aligned}$$

Since, $b$, $c$ and $n$ are integers, the above inequality can be written as:

$$b \geqslant \left\lceil \frac{n - c}{2} \right\rceil \tag{4}$$

Since, $0 > a$ and $b \leqslant c$

$$\begin{aligned} 0 + b + b &< a + b + c \\ 2b &< n \\ b &< \frac{n}{2} \end{aligned}$$

Since, $b$ and $n$ are integers, the above inequality can be written as:

$$b \leqslant \left\lceil \frac{n}{2} \right\rceil - 1 \tag{5}$$

Also, $b \leqslant c$. From this, (3), (4) and (5), we get:

$$\left\lceil \frac{n - c}{2} \right\rceil \leqslant b \leqslant c \tag{6}$$

From (3) and (6), we see that the total number of solutions is:

$$\begin{aligned} &\sum_{c=\lceil \frac{n}{3} \rceil}^{\lceil \frac{n}{2} \rceil - 1} \sum_{b=\lceil \frac{n-c}{2} \rceil}^{c} 1 \\ &= \sum_{c=\lceil \frac{n}{3} \rceil}^{\lceil \frac{n}{2} \rceil - 1} \left( 1 + c - \left\lceil \frac{n - c}{2} \right\rceil \right) \end{aligned}$$

□