INTEGER TRIANGLES WITH FIXED LONGEST SIDE

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Theorem. *The number of solutions for $a+b>c$, where a, b and c are three integers with c already given and $0<a\leqslant b\leqslant c$, is:*

$$f(c)=\begin{cases}\frac{c(c+2)}{4} & \text{if } c\equiv 0 \pmod 2\\ \frac{(c+1)^2}{4} & \text{if } c\equiv 1 \pmod 2\end{cases}$$

Proof. b must be greater than $\frac{c}{2}$ otherwise a will become greater than b. This is shown below.

$$\begin{aligned} c &< a+b \leqslant b+b\\ 2b &> c\\ b &> \frac{c}{2}\end{aligned}$$

Since b and c are integers, the above inequality can also be expressed as:

$$b \geqslant \left\lfloor \frac{c}{2} \right\rfloor + 1 \tag{1}$$

Also,

$$b \leqslant c \tag{2}$$

From (1) and (2), we get:

$$\left\lfloor \frac{c}{2} \right\rfloor + 1 \leqslant b \leqslant c \tag{3}$$

Therefore, there are: $c-\left\lfloor \frac{c}{2} \right\rfloor$ possible values for b. For each value of b, a should satisfy the following inequality:

$$\begin{aligned} a+b &> c\\ a &> c-b\end{aligned}$$

Since, a, b and c are an integers, the above inequality can be written as:

$$a > c - b + 1 \tag{4}$$

Also,

$$0 < a \leqslant b \tag{5}$$

From (4) and (5), we see, that a should satisfy the following inequality for each value of b:

$$c - b + 1 \leqslant a \leqslant b \tag{6}$$

From (3) and (6), we see that the total number of solutions are:

$$\begin{aligned}
f(c) &= \sum_{b=\left\lfloor \frac{c}{2} \right\rfloor+1}^{c} \sum_{a=c-b+1}^{b} 1 \\
&= \sum_{b=\left\lfloor \frac{c}{2} \right\rfloor+1}^{c} (2b - c) \\
&= 2 \sum_{b=\left\lfloor \frac{c}{2} \right\rfloor+1}^{c} b \ - \ c\left(c - \left\lfloor \frac{c}{2} \right\rfloor\right) \\
&= \left(c - \left\lfloor \frac{c}{2} \right\rfloor\right)\left(c + \left\lfloor \frac{c}{2} \right\rfloor + 1\right) - c\left(c - \left\lfloor \frac{c}{2} \right\rfloor\right) \\
&= \left(c - \left\lfloor \frac{c}{2} \right\rfloor\right)\left(\left\lfloor \frac{c}{2} \right\rfloor + 1\right)
\end{aligned}$$

So,

$$f(c) = \left(1 + \left\lfloor \frac{c}{2} \right\rfloor\right)\left(c - \left\lfloor \frac{c}{2} \right\rfloor\right) \tag{7}$$

If $c \equiv 0 \pmod 2$, $\left\lfloor \frac{c}{2} \right\rfloor = \frac{c}{2}$, therefore,

$$\begin{aligned}
f(c) &= \left(1 + \frac{c}{2}\right)\left(c - \frac{c}{2}\right) \\
&= \frac{c(c+2)}{4}
\end{aligned}$$

If $c \equiv 1 \pmod 2$, $\left\lfloor \frac{c}{2} \right\rfloor = \frac{c-1}{2}$, therefore,

$$\begin{aligned}
f(c) &= \left(1 + \frac{c-1}{2}\right)\left(c - \frac{c-1}{2}\right) \\
&= \frac{(c+1)^2}{4}
\end{aligned}$$

□