FACTORS

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1. Representation

A positive integer N can be expressed as a canonical form $p_1^{k_1}.p_2^{k_2}\ldots p_n^{k_n}$, where $p_1,\ p_2,\ \ldots,\ p_n$ are prime numbers and $k_1,\ k_2,\ \ldots,\ k_n$ are positive integers. This representation of an integer will be used in this document.

2. Total number of factors of a number

Theorem. *The number of possible factors of N is $\prod_{i=1}^{n}(k_i+1)$.*

Proof. Any factor of N can be expressed as:

$$p_1^{r_1}.p_2^{r_2}\ldots p_n^{r_n},\ \text{where } 0 \leqslant r_i \leqslant k_i$$

We can raise p_i by $k_i + 1$ different powers for integers $1 \leqslant i \leqslant n$. Thus, the total number of possible factors is:

$$(k_1+1)(k_2+1)\ldots(k_n+1)$$

Hence, the total number of factors of N is:

$$\prod_{i=1}^{n}(k_i+1)$$

□

3. Sum of all factors of a number

Theorem. *The sum of all factors of a positive integer N is:*

$$\prod_{i=1}^{n} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Proof. The sum of all factors of N can be obtained as:

$$(p_1^0 + p_1^1 + \cdots + p_1^{k_1})(p_2^0 + p_2^1 + \cdots + p_2^{k_2}) \ldots (p_n^0 + p_n^1 + \cdots + p_n^{k_n})$$

It can be seen that this is a sum of $\prod_{i=1}^{n}(k_i + 1)$ terms where each term is a distinct factor of N. Also, any factor,

$$p_1^{r_1} . p_2^{r_2} \ldots p_n^{r_n}, \text{ where } 0 \leqslant r_i \leqslant k_i$$

can be obtained in the sum as the term obtained from picking up each power of a prime, from the sum of the series of the powers of that particular prime, and multiplying them.

We observe that the sum of each series of possible powers of a prime is a sum of numbers in geometric progression. Thus, the expression of the sum of all factors of N can be simplified as:

$$(p_1^0 + p_1^1 + \cdots + p_1^{k_1})(p_2^0 + p_2^1 + \cdots + p_2^{k_2}) \ldots (p_n^0 + p_n^1 + \cdots + p_n^{k_n})$$

$$= \prod_{i=1}^{n} \sum_{j=0}^{k_i} p_i^j$$

$$= \prod_{i=1}^{n} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

□

4. Product of all factors of a number

Theorem. *The product of all factors of a positive integer N is:*

$$\sqrt{N}^{\prod_{i=1}^{n}(k_i+1)}$$

Proof. The product of all factors of N can be represented as a positive integer $M = p_1^{q_1}.p_2^{q_2}\dots p_n^{q_n}$, where $p_1, p_2, \dots, p_n$ are prime numbers and $q_1, q_2, \dots, q_n$ are positive integers.

Consider the integer, $N' = \frac{N}{p_r^{k_r}}$. N', which does not have p_r as a factor. The total number of factors of N' is $\prod_{i=j,j\neq r}^{n}(k_j+1)$, which we will call as f.

Each power of p_r which is a factor of N, i.e., $p_r^0, p_r^1, \dots, p_r^{k_r}$, can be multiplied with these $\prod_{i=1,i\neq r}^{n}(k_i+1)$ factors of N' to create a factor of N. So,

$$p_r^{q_r} = p_r^{0f}.p_r^{1f}.\dots.p_r^{k_r f}$$

So,

$$\begin{aligned}
q_r &= (0+1+\dots+k_r)f \\
&= \frac{(k_r)(k_r+1)}{2}f \\
&= \frac{(k_r)(k_r+1)}{2}\prod_{j=1,j\neq r}^{n}(k_j+1) \\
&= \frac{k_r}{2}\prod_{j=1}^{n}(k_j+1)
\end{aligned}$$

Hence,

$$\begin{aligned}
M &= \prod_{i=1}^{n} p_i^{\frac{k_i}{2}\prod_{j=1}^{n}(k_j+1)} \\
&= \sqrt{N}^{\prod_{i=1}^{n}(k_i+1)}
\end{aligned}$$

□